# Secondary Characteristic Classes and Cyclic Cohomology of Hopf Algebras

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## Abstract

Let $X$ be a manifold on which discrete (pseudo)group of diffeomorphisms $\Gamma$ acts, and let $E$ be a $\Gamma$-equivariant vector bundle on $X$. We give a construction of cyclic cocycles on the cross product algebra $C_0^\infty(X) \rtimes \Gamma$ representing the equivariant characteristic classes of $E$. Our formulas generalize Connes' Godbillon-Vey cyclic cocycle. An essential tool of our construction is Connes-Moscovici's theory of cyclic cohomology of Hopf algebras.

## 1   Introduction

In the paper [3] A. Connes provided an explicit construction of the Godbillon-Vey cocycle in the cyclic cohomology. The goal of this paper is to give a similar construction for the higher secondary classes.

First, let us recall Connes' construction. Let $M$ be a smooth oriented manifold and let $\Gamma \subset \mathrm{Diff}^+(M)$ be a discrete group of orientation-preserving diffeomorphisms of $M$. Let $\omega$ be a volume form on $M$. Define the following function on $M \times \Gamma$: $\delta(g) = \frac{\omega^g}{\omega}$, where the superscript denotes the group action. Then one can define one-parametric group of diffeomorphisms of the

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algebra $\mathcal{A} = C_0^\infty(M) \rtimes \Gamma$ by

$$\phi_t(aU_g) = a\delta(g)^t U_g \tag{1.1}$$

This is Tomita-Takesaki group of automorphisms, associated to the weight on $\mathcal{A}$ given by $\omega$.

Consider now the transverse fundamental class – cyclic $q$-cocycle on $\mathcal{A}$ given by

$$\tau(a_0 U_{g_0}, a_1 U_{g_1}, \dots, a_q U_{g_q}) =$$
$$\begin{cases} \frac{1}{q!} \int\limits_M a_0 da_1^{g_0} da_2^{g_0 g_1} \dots da_q^{g_0 g_1 \dots g_{q-1}} & \text{if } g_0 g_1 \dots g_q = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

To study the behavior of this cocycle under the 1-parametric group (1.1), consider the "Lie derivative" $\mathcal{L}$ acting on the cyclic complex by $\mathcal{L}\xi = \frac{d}{dt}|_{t=0}\phi_t^*\xi$, $xi$ being a cyclic cochain. It turns out that in general $\tau$ is not invariant under the group (1.1), and $\mathcal{L}\tau \neq 0$.

However, it was noted by Connes that one always has

$$\mathcal{L}^{q+1}\tau = 0 \tag{1.3}$$

and $\mathcal{L}^q\tau$ is invariant under the action of the group (1.1). One deduces from this that if $\iota_\delta$ is the analogue of the interior derivative (see [3]), then $\iota_\delta \mathcal{L}^q\tau$ is a cyclic cocycle.

This is Connes' Godbillon-Vey cocycle. It can be related to the Godbillon-Vey class as follows. Let $[GV] \in H^*(M_\Gamma)$ be the Godbillon-Vey class, where $M_\Gamma = M \times_\Gamma E\Gamma$ is the homotopy quotient. Connes defines canonical map $\Phi : H^*(M_\Gamma) \to HP^*(\mathcal{A})$. Then one has

$$\Phi([GV]) = [\iota_\delta \mathcal{L}^q\tau] \tag{1.4}$$

The class of this cocycle is independent of the choice of the volume form. To prove this one can use Connes' noncommutative Radon-Nicodym theorem to conclude that if one changes the volume form, the one-parametric group $\phi_t$ remains the same modulo inner automorphisms.

A natural problem then is to extend this construction to the cocycles corresponding to the other secondary characteristic classes. It was noted by Connes [4] that if instead of 1-dimensional bundle of $q$-forms on $M$ one

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considers $\Gamma$ equivariant trivial bundle of rank $n$, then in place of 1-parametric group (1.1) one encounters coaction of the group $GL_n(\mathbb{R})$ on the algebra $\mathcal{A}$. The difficulty is that for $n > 1$ this group is not commutative, and one can not replace this coaction by the action of the dual group, similarly to (1.1).

In this paper we show that Connes-Moscovici theory of cyclic cohomology for Hopf algebras (cf [5, 6]) provides a natural framework for the higher-dimensional situation and allows one to give construction of the secondary characteristic cocycles.

The situation we consider is the following. We have an orientation-preserving action of discrete group (or pseudogroup) $\Gamma$ on the oriented manifold $M$, and a trivial bundle $E$ on $M$ equivariant with respect to this action. Well-known examples in which such a situation arises are the following (cf. [4], [13],[5]). Let $V$ be a manifold on which a discrete group (or pseudogroup) $G$ acts, and let be $E_0$ a bundle (not necessarily trivial) on $V$, equivariant with respect to the action of $G$. Let $U_i$, $i \in I$, be an open covering of $V$ such that restriction of $F$ on each $U_i$ is trivial. Put $M = \sqcup U_i$, and let $E$ be the pull-back of $E_0$ to $M$ by the natural projection. Then we have an action of the following pseudogroup $\Gamma$ on $M$: $\Gamma = \{g_{i,j} \mid g \in G \, i, j \in I\} \cup id$, where $\mathrm{Dom}\, g_{i,j} = g^{-1}(U_j) \cap U_i \subset U_i$, $\mathrm{Ran}\, g_{i,j} = g(U_i) \cap U_j \subset U_j$, and the natural composition rules. The bundle $E$ is clearly equivariant with respect to this action. Our construction, described below, provides classes in the cyclic cohomology of the cross-product algebra $C_0^\infty(M) \rtimes \Gamma$, rather than in the cyclic cohomology of $C_0^\infty(V) \rtimes G$. However, cross-product algebras $C_0^\infty(M) \rtimes \Gamma$ and $C_0^\infty(V) \rtimes G$ are Morita equivalent, and hence have the same cyclic cohomology.

Another natural example is provided by the manifold $V$ with foliation $F$, and a bundle $E_0$ which is holonomy equivariant. We can always choose (possibly disconnected) complete transversal $M$, such that restriction $E$ of $E_0$ to $M$ is trivial. Let $\Gamma$ be the holonomy pseudogroup acting on $M$. $E$ is clearly equivariant with respect to this action. In this case again the cross-product algebra $C_0^\infty(M) \rtimes \Gamma$ is Morita equivalent to the full algebra of the foliation $C_0^\infty(V, F)$.

We construct then a map from the cohomology of the truncated Weil algebra (cf. e.g. [12]) $W(\mathfrak{g}, O_n)_q$ to the periodic cyclic cohomology $HP^*(\mathcal{A})$ of the algebra $\mathcal{A} = C_0^\infty(M) \rtimes \Gamma$. The construction is the following. We consider the action of the *differential graded* Hopf algebra $\mathcal{H}(GL_n(\mathbb{R}))$ of differential forms on the group $GL_n(\mathbb{R})$ on the differential graded algebra $\Omega_0^*(M) \rtimes \Gamma$, where $\Omega_0^*(M)$ denotes the algebra of compactly supported differential forms

on $M$. The use of differential graded algebras allows one to conveniently encode different identities, similar to (1.3). We then show that Connes-Moscovici theory (or rather differential graded version of it) allows one to define a map from the cyclic complex of $\mathcal{H}(GL_n(\mathbb{R}))$ to the cyclic complex of $\Omega_0^*(M) \rtimes \Gamma$. We then relate cyclic complex of $\mathcal{H}(GL_n(\mathbb{R}))$ to the Weil algebra, and cyclic cohomology of $\Omega_0^*(M) \rtimes \Gamma$ to the cyclic cohomology of $\mathcal{A}$.

The paper is organized as follows. In the next two sections we discuss cyclic complexes for differential graded algebras and differential graded Hopf algebras respectively. In the section 4 we show that two different Hopf actions, which coincide "modulo inner automorphisms" induce the same Connes-Moscovici characteristic map in cyclic cohomology, and discuss some other properties of the characteristic map. In the section 5 we construct the action of $\mathcal{H}(GL_n(\mathbb{R}))$ on $\Omega_0^*(M) \rtimes \Gamma$. In the section 6 we relate cyclic complex of the Hopf algebra $\mathcal{H}(GL_n(\mathbb{R}))$ with the Weil algebras. Finally, in the section 7 we prove an analogue of the formula (1.4) for the cocycles we construct.

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## 2 Cyclic complex for differential graded algebras

In this section we collect some preliminary standard facts about cyclic cohomology of the differential graded algebras, and give cohomological version of some results of [9].

Recall that the cyclic module $X^*$ is given by the cosimplicial module with the face maps $\delta_i : X^{n-1} \to X^n$ and and degeneracy maps $\sigma_i : X^n \to X^{n-1}$ $0 \leq i \leq n$, satisfying the usual axioms. In addition, we have for each $n$ an action of $\mathbb{Z}_{n+1}$ on $X^n$, with the generator $\tau_n$ satisfying

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \text{ for } 1 \leq i \leq n \text{ and } \tau_n \delta_0 = \delta_n \tag{2.1}$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \text{ for } 1 \leq i \leq n \text{ and } \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \tag{2.2}$$

$$\tau_n^{n+1} = id \tag{2.3}$$

For every cyclic object one can construct operators $b : X^n \to X^{n+1}$ and $B : X^n \to X^{n-1}$, defined by the formulas

$$b = \sum_{j=0}^{n}(-1)^j d_j \tag{2.4}$$

$$B = \left(\sum_{j=0}^{n-1}(-1)^{j(n-1)}\tau_{n-1}^j\right)\sigma_{n+1}(1-(-1)^{n-1}\tau_n) \tag{2.5}$$

where

$$\sigma_{n+1} = \sigma_n \tau_{n+1} \tag{2.6}$$

These operators satisfy

$$b^2 = 0 \tag{2.7}$$
$$B^2 = 0 \tag{2.8}$$
$$bB + Bb = 0 \tag{2.9}$$

Hence for any cyclic object $X^*$ we can construct a bicomplex $\mathcal{B}^{*,*}(X)$ as follows: $\mathcal{B}^{p,q}$, $p$, $q \geq 0$ is $X^{p-q}$, or 0 if $p < q$, and the differential $\mathcal{B}^{p,q} \to \mathcal{B}^{p+1,q}$ (resp. $\mathcal{B}^{p,q+1}$) is given by $b$ (resp. $B$). By removing restriction $p$, $q \geq 0$ we obtain periodic bicomplex $\mathcal{B}_{\text{per}}$. Notice that it has periodicity induced by the tautological shift $S : \mathcal{B}_{\text{per}}^{p,q} \to \mathcal{B}_{\text{per}}^{p+1,q+1}$.

Let now $\Omega^*$ be a unital graded (DG) algebra, positively graded. We can associate with it a cyclic object as follows (the differential $d$ is not used in this definition).

Let $C^k(\Omega^*)$ be the space of continuous $k+1$-linear functionals on $\Omega^*$. The face and degeneracy maps are given by

$$(\delta_j\phi)(a_0, a_1, \ldots, a_{k+1}) = \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{k+1}) \text{ for } 0 \leq j \leq n-1$$
$$(\delta_n\phi)(a_0, a_1, \ldots, a_{k+1}) = (-1)^{\deg a_{k+1}(\deg a_0 + \cdots + \deg a_k)}\phi(a_{k+1}a_0, a_1, \ldots a_k) \tag{2.10}$$

$$(\sigma_j\phi)(a_0, \ldots, a_{k-1}) = \phi(a_0, \ldots, a_j, 1, a_{j+1}, \ldots a_{k-1}) \tag{2.11}$$

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and the cyclic action is given by

$$(\tau_k \phi)(a_0, \ldots, a_k) = (-1)^{\deg a_k (\deg a_0 + \ldots \deg a_{k-1})} \phi(a_k, a_0, \ldots, a_{k-1}) \qquad (2.12)$$

Cohomology of total complex of the bicomplex $\mathcal{B}$ (resp. $\mathcal{B}_{\mathrm{per}}$) where we consider only *finite* cochains, is the cyclic (resp. periodic cyclic) cohomology of $\Omega^*$, for which we use notation $HC^*(\Omega^*)$ (resp. $HP^*(\Omega^*)$).

Suppose now that $\Omega^*$ is a differential graded (DG) algebra with the differential of degree 1. We say that $\phi \in C^k(\Omega^*)$ has weight $m$ if $\phi(a_0, a_1, \ldots, a_k) = 0$ unless $\deg a_0 + \deg a_1 + \cdots + \deg a_k = m$. We denote by $C^{k,p}(\Omega^*) \subset C^k(\Omega^*)$ set of weight $-p$ functionals. Notice that in this case each $C^k(\Omega^*)$ is a complex in its own right, with the grading defined above and the differential $(-1)^k d$, where we extend $d$ to $C^k(\Omega^*)$ by

$$d\phi(a_0, a_1, \ldots, a_k) = \sum_{j=0}^{k} (-1)^{\deg a_0 + \ldots \deg a_{j-1}} \phi(a_0, \ldots, da_j, \ldots, a_k) \qquad (2.13)$$

Then $db - bd = 0$, $dB - Bd = 0$, and hence in this situation $\mathcal{B}$ and $\mathcal{B}_{\mathrm{per}}$ become actually tricomplexes. Cyclic (resp. periodic cyclic) cohomology of the DG algebra $(\Omega^*, d)$ is then defined as the cohomology of the total complex of the tricomplex $\mathcal{B}$ (resp. $\mathcal{B}_{\mathrm{per}}$) where we consider only *finite* cochains. Notations for the cyclic and periodic cyclic cohomologies are $HC^*((\Omega^*, d))$ and $HP^*((\Omega^*, d))$.

One can show that cyclic cohomology can be computed by the *normalized* complex, i.e. one where cochains satisfy

$$\phi(a_0, a_1, \ldots, a_k) = 0 \text{ if } a_i = 1, \ i \geq 1 \qquad (2.14)$$

We will need the following result about the cyclic cohomology.

**Theorem 1.** *Let $\mathcal{A} = \Omega^0$ be the 0-degree part of $\Omega^*$ (which we consider as a trivially graded algebra with the zero differential). We then have a natural map of (total) complexes $I : \mathcal{B}_{per}(\mathcal{A}) \to \mathcal{B}_{per}((\Omega^*, d))$ (extension of polilinear forms by 0 to $\Omega^*$). Then the induced map in cohomology is an isomorphism. $HP^*(\mathcal{A}) \to HP^*((\Omega^*, d))$.*

To prove the theorem and to write an explicit formula for the map $R : \mathcal{B}_{\mathrm{per}}((\Omega^*, d)) \to \mathcal{B}_{\mathrm{per}}(\mathcal{A})$, inducing the inverse isomorphism in the periodic cyclic cohomology, we need the following fact (Rinehart formula) which we use as stated in [9].

Let $D$ be a derivation of the graded algebra $\Omega^*$ of degree $\deg D$, i.e. a linear map $D : \Omega^* \to \Omega^{*+\deg D}$ satisfying

$$D(ab) = (Da)b + (-1)^{\deg D \deg a} a D(b) \tag{2.15}$$

It defines an operator on the complex $\mathcal{B}(\Omega^*)$, by

$$\mathcal{L}_D \phi(a_0, a_1, \ldots, a_k) = \sum_{i=0}^{k} (-1)^{\deg D(a_0 + \cdots + a_{i-1})} \phi(a_0, \ldots, D(a_i), \ldots, a_k) \tag{2.16}$$

which commutes with $b$, $B$. The action of this operator on the periodic cyclic bicomplex is homotopic to zero, with the homotopy constructed as follows. Define operators $e_D : C^{k-1}(\Omega^*) \to e_D : C^k(\Omega^*)$, $E_D : e_D : C^{k+1}(\Omega^*) \to e_D : C^k(\Omega^*)$ by

$$e_D \phi(a_0, a_1, \ldots a_k) = (-1)^{k+1} \phi(D(a_k) a_0, a_1 \ldots, a_{k-1}) \tag{2.17}$$

$$E_D \phi(a_0, a_1, \ldots a_k) = \sum_{1 \le i \le j \le k} (-1)^{ik+1} \phi(1, a_i, a_{i+1}, \ldots, a_{j-1}, D a_j, \ldots, a_k, a_0, \ldots) \tag{2.18}$$

Then

$$[b + B, e_D + E_D] = \mathcal{L}_D \tag{2.19}$$

We now proceed with the proof of the Theorem 2

*Proof of the Theorem 2.* Consider the derivation $D$ of degree 0 given by $Da = (\deg a)a$. On the polilinear form of weight $m$ $D$ acts by $m$. Define the homotopy $h$ to be $\frac{1}{m}(e_D + E_D)$ on the forms of weight $m > 0$ and 0 on the forms of weight 0. We define map of complexes $R : \mathcal{B}_{\mathrm{per}}((\Omega^*, d)) \to \mathcal{B}_{\mathrm{per}}(\mathcal{A})$ by

$$R\phi = c_{k,m}(dh)^m \phi \text{ for } \phi \in C^{k,m} \tag{2.20}$$

where

$$c_{k,m} = (-1)^{km + \frac{m^2 - m}{2}} \tag{2.21}$$

This is a map of (total) complexes. Indeed, using identities $(b+B)h + h(b+B) = id$ and $(b+B)d - d(b+B) = 0$ we have:

$$(b+B)R\phi - R(b+B+(-1)^k d)\phi =$$
$$c_{k,m}\left(((b+B)(dh)^m - (-1)^m(dh)^m(b+B)) + (-1)^m(dh)^{m-1}d\right)\phi =$$
$$c_{k,m}\left(-(-1)^m(dh)^{m-1}d + (-1)^m(dh)^{m-1}d\right)\phi = 0 \quad (2.22)$$

It is clear that

$$R \circ I = id \quad (2.23)$$

As for $I \circ R$ we have

$$I \circ R = id - (\partial \circ H + H \circ \partial) \quad (2.24)$$

where $\partial = b + B \pm d$ – total differential in the complex $\mathcal{B}_{per}\left((\Omega^*, d)\right)$, $\pm d$ being $(-1)^k d$ on $C^{k,m}$, and the homotopy $H$ is given by the formula

$$H\phi = \sum_{j=0}^{m-1} c_{k,j} h(dh)^j \phi \text{ for } \phi \in C^{k,m} \quad (2.25)$$

This equality is also verified by direct computation.

Indeed, we have, for actions on $C^{k,m}$

$$H \circ (b+B) = \sum_{j=0}^{m-1} c_{k,j}((dh)^j - (b+B)h(dh)^j) + \sum_{j=1}^{m-1} c_{k,j}(-1)^j(hd)^j \quad (2.26)$$

$$(b+B) \circ H = \sum_{j=0}^{m-1} c_{k,j}(b+B)h(dh)^j \quad (2.27)$$

$$H \circ (\pm d) = \sum_{j=0}^{m-2}(-1)^k c_{k,j}(hd)^{j+1} = \sum_{j=1}^{m-1}(-1)^k c_{k,j-1}(hd)^j \quad (2.28)$$

$$(\pm d) \circ H = \sum_{j=0}^{m-1}(-1)^{k-j-1} c_{k,j}(dh)^{j+1} = \sum_{j=1}^{m-1}(-1)^{k-j} c_{k,j-1}(dh)^j \quad (2.29)$$

and adding these equalities we get the desired result. $\square$

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# 3 Cyclic complex for differential graded Hopf algebras

In this section we reproduce Connes-Moscovici's construction of the cyclic module of a Hopf algebra (cf. [5, 6]) in the differential graded context.

Let us start with the graded Hopf algebra $\mathcal{H}^*$. We need to fix a modular pair, i.e. a homomorphism $\delta : \mathcal{H}^* \to \mathbb{C}$ and a group-like element $\sigma \in \mathcal{H}^0$. Using the standard notations for the coproduct and antipode, define the twisted antipode $\widetilde{S}_\delta$ by

$$\widetilde{S}_\delta(h) = \sum S(h_{(0)})\delta(h_{(1)}) \tag{3.1}$$

Suppose that the following condition holds:

$$\left(\sigma^{-1}\widetilde{S}_\delta\right)^2 = id \tag{3.2}$$

Then Connes and Moscovici show that one can define a cyclic object $(\mathcal{H}^*)^\sharp = \left\{(\mathcal{H}^*)^{\otimes n}\right\}_{n \geq 1}$ as follows. Face and degeneracy operators are given by

$$\delta_0(h^1 \otimes \ldots \otimes h^{n-1}) = 1 \otimes h^1 \otimes \ldots \otimes h^{n-1}$$
$$\delta_j(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^n \text{ for } 1 \leq j \leq n-1,$$
$$\delta_n(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma$$
$$\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} \tag{3.3}$$

The cyclic operators are given by

$$\tau_n(h^1 \otimes \ldots \otimes h^{n+1}) =$$
$$\sum (-1)^{\sum\limits_{j > i \geq 0} \deg h^1_i \deg h^j} \left(\widetilde{S}h^1\right)_{(0)} h^2 \otimes \cdots \otimes \left(\widetilde{S}h^1\right)_{(n-2)} h^n \otimes \left(\widetilde{S}h^1\right)_{(n-1)} \sigma \tag{3.4}$$

where

$$\left(\Delta^{n-1}\widetilde{S}h^1\right) = \sum \left(\widetilde{S}h^1\right)_{(0)} \otimes \ldots \left(\widetilde{S}h^1\right)_{(n-1)}$$

It is verified in [6] that the operations above indeed define a structure of a cyclic module.

Hence we can define cyclic and periodic cyclic complexes of this cyclic module. Suppose now that our Hopf algebra $\mathcal{H}^*$ is a DG Hopf algebra with the differential $d$ of degree 1. Then complexes $\mathcal{B}$ and $\mathcal{B}_{\text{per}}$ have an extra differential defined to be $(-1)^n d$ on $(\mathcal{H}^*)^{\otimes n}$ where we extend $d$ by

$$d(h^1 \otimes h^2 \cdots \otimes h^n) = \sum_{i=1}^{n} (-1)^{\deg h^1 + \dots \deg h^{i-1}} h^1 \otimes h^2 \dots dh^i \cdots \otimes h^n \quad (3.5)$$

We consider the total complexes of the *finite* cochains in the resulting tricomplexes, and define cyclic and peridic cyclic cohomology of DG Hopf algebra as cohomology of these complexes.

Suppose now that we are given an action $\pi$ of a differential graded Hopf algebra $\mathcal{H}^*$ on $\Omega^*$, which agrees with the differential graded structures on $\mathcal{H}^*$ and $\Omega^*$, i.e. in addition to the general properties of Hopf algebra action we have

$$\deg \pi(h)(a) = \deg h + \deg a \quad (3.6)$$

$$d\left(\pi(h)(a)\right) = \pi(dh)(a) + (-1)^{\deg h} \pi(h)(da) \quad (3.7)$$

where $h \in \mathcal{H}^*$, $a \in \Omega^*$. We will often omit $\pi$ from our notations and write just $h(a)$ if it is clear what action we are talking about.

Suppose that $\int$ is a closed graded $\sigma$-trace on $\Omega^*$, $\delta$-invariant under the action of $\mathcal{H}^*$, i.e.

$$\int \pi(h)(a)b = \int a\pi\left(\widetilde{Sh}\right)b \quad (3.8)$$

$$\int ab = \int b\pi(\sigma)(a) \quad (3.9)$$

Then one has a map of cyclic modules $\chi_\pi : (\mathcal{H}^*)^\sharp \to (\Omega^*)^\sharp$, given by

$$\chi_\pi(h^1 \otimes h^2 \cdots \otimes h^k)(a_0, a_1, \dots a_k) = \lambda \int a_0 \pi(h^1)(a_1) \dots \pi(h^k)(a_k) \quad (3.10)$$

where

$$\lambda = (-1)^{\sum\limits_{j > i \geq 0} \deg h^j \deg a_i}$$

This map also commutes with the differential $d$, and hence induces a characteristic map $\chi_\pi : \mathcal{B}(\mathcal{H}^*, d) \to \mathcal{B}(\Omega^*, d)$, as well as corresponding maps in cohomology.

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# 4 Properties of the characteristic map

We will consider now two actions of $\mathcal{H}^*$ on $\Omega^*$ which are conjugated by the inner automorphism. We will work in the assumption that $\Omega^*$ is unital, indicating the changes which need to be made in the nonunital case in the Remark 7. More precisely, let $\rho^+$ and $\rho^-$ be two degree-preserving linear maps from $\mathcal{H}^*$ to $\Omega^*$, which commute with the differentials. We suppose that they are inverse to each other with respect to convolution:

$$\sum \rho^+(h_{(0)})\rho^-(h_{(1)}) = \varepsilon(h)1 \tag{4.1}$$

and satisfy cocycle identities:

$$\rho^+(hg) = \sum \rho^+(h_{(0)})\pi(h_{(1)})(\rho^+(g)) \tag{4.2}$$

$$\rho^-(gh) = \sum \pi(h_{(0)})(\rho^-(g))\rho^-(h_{(1)}) \tag{4.3}$$

$$\rho^+(1) = \rho^-(1) = \rho^+(\sigma) = \rho^-(\sigma) = 1 \tag{4.4}$$

Then one can define a new action $\pi'$ of $\mathcal{H}^*$ on $\Omega^*$ by

$$\pi'(h)(a) = \sum (-1)^{\deg h_{(2)} \deg a} \rho^+(h_{(0)})\pi(h_{(1)})(a)\rho^-(h_{(2)}) \tag{4.5}$$

**Lemma 2.** *Equation* (4.5) *defines an action of the DG Hopf algebra* $\mathcal{H}^*$ *on the DG algebra* $\Omega^*$.

*Proof.* We check all the required properties. First

$$\pi'(h)(ab) = \sum (-1)^{\deg ab \deg h_{(2)}} \rho^+(h_{(0)})\pi(h_{(1)})(ab)\rho^-(h_{(2)}) =$$

$$\sum (-1)^{(\deg a + \deg b)\deg h_{(3)}} (-1)^{\deg a \deg h_{(2)}} \rho^+(h_{(0)})\pi(h_{(1)})(a)\pi(h_{(2)})(b)\rho^-(h_{(3)}) =$$

$$\sum (-1)^{(\deg a + \deg b)\deg h_{(5)}} (-1)^{\deg a(\deg h_{(2)} + \deg h_{(3)} + \deg h_{(4)})}$$

$$\rho^+(h_{(0)})\pi(h_{(1)})(a)\rho^-(h_{(2)})\rho^+ h_{(3)}\pi(h_{(4)})(b)\rho^-(h_{(5)}) =$$

$$\sum (-1)^{\deg a \deg h_{(1)}} \pi'(h_{(0)})(a)\pi'(h_{(1)})(b) \tag{4.6}$$

Then

$$
\pi'(hg)(a) =
$$
$$
\sum(-1)^{\deg a(\deg h_{(2)}+\deg g_{(3)})}\rho^+(h_{(0)}g_{(0)})\pi(h_{(1)}g_{(1)})(a)\rho^-(h_{(2)}g_{(2)}) =
$$
$$
\sum(-1)^{\deg a(\deg h_{(3)}+\deg g_{(2)}+\deg h_{(4)})}
$$
$$
\rho^+(h_{(0)})\pi(h_{(1)})(\rho^+(g_{(0)}))\pi(h_{(2)}g_{(1)})(a)\pi(h_{(3)})\rho^-(g_{(2)})\rho^-(h_{(4)}) =
$$
$$
\pi'(h)\left(\pi'(g)(a)\right) \quad (4.7)
$$

Also

$$
\pi'(h)(1) = \sum \rho^+(h_{(0)})h_{(1)}(1)\rho^-(h_{(3)}) = \varepsilon(h)1 \qquad (4.8)
$$
$$
\pi'(1)(a) = \rho^+(1)a\rho^-(1) = a \qquad (4.9)
$$

and

$$
d\left(\pi'(h)(a)\right) = d\sum(-1)^{\deg a \deg h_{(3)}}\rho^+(h_{(0)})\pi(h_{(1)})(a)\rho^-(h_{(2)}) =
$$
$$
\sum(-1)^{\deg a \deg h_{(3)}}\rho^+(dh_{(0)})\pi(h_{(1)})(a)\rho^-(h_{(2)}) +
$$
$$
\sum(-1)^{\deg a \deg h_{(3)}+\deg h_{(0)}}
$$
$$
\rho^+(h_{(0)})\left(\pi(dh_{(1)})(a)+(-1)^{\deg h_{(1)}}\pi(h_{(1)})(da)\right)\rho^-(h_{(2)}) +
$$
$$
\sum(-1)^{\deg a \deg h_{(3)}+\deg h_{(0)}+\deg h_{(1)}+\deg a}\rho^+(h_{(0)})\pi(h_{(1)})(a)\rho^-(dh_{(2)}) =
$$
$$
\pi'(dh)(a)+(-1)^{\deg h}\pi'(h)(da) \quad (4.10)
$$

$\square$

Suppose now that $\oint$ is the closed $\delta$-invariant $\sigma$-trace for both actions $\pi$ and $\pi'$. In this case we have two characteristic maps $\chi_\pi$ and $\chi_{\pi'}$ from $\mathcal{B}(\mathcal{H}^*, d)$ to $\mathcal{B}(\Omega^*, d)$. Then we have the following

**Proposition 3.** *Let $\pi$ and $\pi'$ be two actions of $\mathcal{H}^*$ on $\Omega^*$, conjugated by inner automorphisms, and suppose that they both satisfy conditions (3.8),(3.9). Let $\chi_\pi$, $\chi_{\pi'}$ be the corresponding characteristic maps. Then induced maps in cohomology $HC^*(\mathcal{H}^*, d) \to HC^*(\Omega^*)$ coincide.*

*Proof.* Let $M_2(\Omega^*) = \Omega^* \otimes M_2(\mathbb{C})$ be the differential graded algebra of $2 \times 2$ matrices over the algebra $\Omega^*$. We can define an action $\pi_2$ of $\mathcal{H}^*$ on $\Omega^* \otimes M_2(\mathbb{C})$

by $\pi_2(h)(a \otimes m) = \pi(h)(a) \otimes m$, where $h \in \mathcal{H}^*$, $a \in \Omega^*$, $m \in M_2(\mathbb{C})$. Put now

$$\rho_2^+(h) = \begin{pmatrix} \rho^+(h) & 0 \\ 0 & \varepsilon(h) \end{pmatrix} \quad \rho_2^-(h) = \begin{pmatrix} \rho^-(h) & 0 \\ 0 & \varepsilon(h) \end{pmatrix} \tag{4.11}$$

It is easy to see that $\rho_2^+$, $\rho_2^-$ satisfy equations (4.1)-(4.4), and hence we can twist the action $\pi_2$ by $\rho_2^+$, $\rho_2^-$ to define a new action $\pi_2'$, as in (4.5).

Consider now the linear functional $\fint_2$ on $M_2(\Omega^*)$ defined by

$$\fint_2 (a \otimes m) = \left(\fint a\right) (\operatorname{tr} m) \tag{4.12}$$

Then $\fint_2$ is a closed graded $\delta$-invariant $\sigma$-trace on $M_2(\Omega^*)$ with respect to the action $\pi_2'$. Hence we can define the characteristic map $\chi_{\pi_2'} : \mathcal{B}(\mathcal{H}^*, d) \to \mathcal{B}(M_2(\Omega^*), d)$

Consider now two imbeddings $i$, $i' : \Omega^* \hookrightarrow M_2(\Omega^*)$ defined by

$$i(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \quad i'(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{4.13}$$

It is easy to see that $i^* \circ \chi_{\pi_2'} = \chi_\pi$ and $(i')^* \circ \chi_{\pi_2'} = \chi_{\pi'}$. Now to finish the proof it is enough to recall the well-known fact that $i$ and $i'$ induce the same map in cyclic cohomology. Since we will later need an explicit homotopy between $\chi_\pi$ and $\chi_{\pi'}$ we give the proof below.

Put $u_t = \exp t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ Put $i_t(a) = u_t i(a) u_t^{-1}$. Notice that $i_0 = i$, $i_{\pi/2} = i'$. Consider the family of maps $i_t^* : \mathcal{B}(M_2(\Omega^*)) \to \mathcal{B}(\Omega^*)$. Since we have $\frac{d}{dt} i_t(a) = [g, i_t(a)]$, where $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ these maps satisfy $\frac{d}{dt} i_t^* = i_t^* L_g$, where $L_g : C^k(M_2(\Omega^*)) \to C^k(M_2(\Omega^*))$ is the operator defined by

$$L_g \phi(x_0, \ldots, x_k) = \sum_{j=0}^{k} \phi(x_0, \ldots, [g, x_j], \ldots, x_k)$$

Define also an operator $I_g : C^k(M_2(\Omega^*)) \to C^{k-1}(M_2(\Omega^*))$ by

$$I_g \phi(x_0, \ldots, x_{k-1}) = \sum_{j=0}^{k-1} \phi(x_0, \ldots, x_j, g, x_{j+1}, \ldots, x_{k-1}) \tag{4.14}$$

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Then it is easy to verify that $[b, I_g] = L_g$, $[B, I_g] = 0$ and $[d, I_g] = 0$. Hence $L_g = \partial I_g + I_g \partial$, $\partial = \pm d + b + B$. We conclude that $i_1^* - i_0^* = K\partial + \partial K$, where the homotopy $K$ is given by $K\phi = \int_0^{\pi/2} i_t^* I_g$.

Hence

$$\chi_{\pi'} - \chi_\pi = \partial H + H\partial \qquad (4.15)$$

where $H = K \circ \chi_{\pi_2'}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now note that the complex $\mathcal{B}(\mathcal{H}^*)$ has a natural weight filtration by subcomplexes $F^l\mathcal{B}(\mathcal{H}^*, d)$, where

$$F^l\mathcal{B}(\mathcal{H}^*, d) = \{\alpha_1 \otimes \alpha_2 \cdots \otimes \alpha_j \mid \deg\alpha_1 + \deg\alpha_2 + \ldots \deg\alpha_j \geq l \quad (4.16)$$

Suppose now that $\displaystyle\int$ has weight $q$, i.e.

$$\int a = 0 \text{ if } \deg a \neq q \qquad (4.17)$$

Notice that in this case $\chi_\pi$ reduces the total degree by $q$ Then following then proposition is clear:

**Proposition 4.** *The characteristic map is $0$ on $F^l\mathcal{B}(\mathcal{H}^*)$ for $l > q$.*

Let $\mathcal{B}(\mathcal{H}^*, d)_l$ denote the truncated cyclic bicomplex:

$$\mathcal{B}(\mathcal{H}^*, d)_l = \mathcal{B}(\mathcal{H}^*, d) / F^{l+1}\mathcal{B}(\mathcal{H}^*, d) \qquad (4.18)$$

Then we immediately have the following

**Corollary 5.** *The characteristic map $\chi_\pi$ defined in* (3.10) *induces the map from the complex $\mathcal{B}(\mathcal{H}^*, d)_q$ to the cyclic complex of the differential graded algebra $\Omega^*$.*

This new map will also be denoted $\chi_\pi$. We use the notation

$$HC^*(\mathcal{H}^*, d)_l = H^*(\mathcal{B}(\mathcal{H}^*, d)_l) \qquad (4.19)$$

for the cohomology of the complex $\mathcal{B}(\mathcal{H}^*, d)_l$. The explicit form of the homotopy in the Proposition 3 now implies the following

**Corollary 6.** *Suppose in addition to the conditions of the Proposition 3 that (4.17) is satisfied. Then the two maps in cohomology induced by $\chi_\pi, \chi_{\pi'}$ : $\mathcal{B}\left(\mathcal{H}^*, d\right)_q \to \mathcal{B}\left(\Omega^*\right)$ are the same.*

*Proof.* We use the notations of the proof of the Proposition 3. There we established that $\chi_{\pi'} - \chi_\pi = \partial H + H \partial$. We need only to verify that $H$ is well defined on the quotient complex $\mathcal{B}\left(\mathcal{H}^*, d\right)_q$. But since $H = K \circ \chi_{\pi_2'}$, and $\chi_{\pi_2'}$ is easily seen to be 0 on $F^{q+1}\mathcal{B}\left(\mathcal{H}^*, d\right)$, the result follows. $\square$

**Remark 7.** We worked above in the assumption that the DG algebra $\Omega^*$ is unital. If this is not the case some changes should be made. First of all, $\rho^+(h)$, $\rho^-(h)$ now don't have to be elements of the algebra, but rather multipliers, such that $\pi'$ defined in (4.5) is a Hopf action. Moreover, we need to require that if $m$ is such a multiplier, then $\int ma = \int am \ \forall a \in \Omega^*$. Then the the Proposition 3 remains true. Characteristic maps in this case take values in the $\mathcal{B}$ complex of the algebra $\Omega^*$ with unit adjoined. The homotopy between two characteristic maps is still given by explicit formula (4.15), which continues to make sense in the nonunital situation. Indeed, it is sufficient to define $I_g \chi_{\pi_2'}$, which can be defined by the same formula as above, provided one treats $g$ and $\pi_2'(h)(g)$ as multipliers of the algebra $M_2\left(\Omega^*\right)$, with $\pi_2'(g)$ defined to be $\begin{pmatrix} 0 & \rho^+(h) \\ -\rho^-(h) & 0 \end{pmatrix}$

Finally, we collect all the information we will need to use in the next sections.

**Theorem 8.** *Let $(\Omega^*, d)$ be a differential graded algebra, and $\int$ a linear functional on $\Omega^*$ of weight $q$, and let $\mathcal{A} = \Omega^0$ be the degree 0 part of $\Omega^*$. Let $\pi$ be an action of the DG Hopf algebra $\mathcal{H}^*$ act on the DGA $\Omega^*$. Suppose that $\int$ is a $\delta$-invariant $\sigma$-trace with respect to $\pi$. Then characteristic map (3.10) defines a map in cohomology $HC^i(\mathcal{H}^*)_q \to HP^{i-q}(\mathcal{A})$. Suppose now that $\pi'$ is another action of $\mathcal{H}^*$ on $\Omega^*$, obtained from $\pi$ by twisting by a cocycle (4.5). Then if $\int$ is a $\delta$-invariant $\sigma$-trace with respect to $\pi'$, the maps in cohomology $HC^i(\mathcal{H}^*)_q \to HP^{i-q}(\mathcal{A})$ induced by $\chi_\pi$, $\chi_{\pi'}$ are the same.*

# 5   Secondary characteristic classes

Let $M$ be a manifold, and let $\Gamma$ be a discrete pseudogroup of diffeomorphisms of $M$, acting from the right.

By this we mean a set $\Gamma$ such that every element $g$ of $\Gamma$ defines a local diffeomorphism of $M$, i.e. diffeomorphism $g : \mathrm{Dom}\, g \to \mathrm{Ran}\, g$, where $\mathrm{Dom}\, g$, $\mathrm{Ran}\, g \subset M$ – open subsets of $M$, and that we have partially defined operations of composition and inverse such that

1. If $g \in \Gamma$ then $g^{-1} : \mathrm{Ran}\, g \to \mathrm{Dom}\, g$ is also in $\Gamma$.

2. If $g_1, g_2 \in \Gamma$ then $g_1 g_2$ with domain $g_1^{-1}\left(\mathrm{Dom}\, g_2 \cap \mathrm{Ran}\, g_1\right)$ and range $g_2\left(\mathrm{Dom}\, g_2 \cap \mathrm{Ran}\, g_1\right)$ is in $\Gamma$.

3. $id : M \to M$ is in $\Gamma$.

Note that we use a wide definition of pseudogroups, and do not include any saturation axioms.

Let $E$ be a trivial vector bundle on $M$, equivariant with respect to the action of $\Gamma$. In other words, every $g \in \Gamma$ defines for every $x \in \mathrm{Dom}\, g$ a linear map $E_x \to E_{xg}$.

For the rest of the paper we suppose the following:

*If $g_1$ and $g_2 \in \Gamma$ are such that they induce the same diffeomorphisms and the same action on the bundle, then $g_1 = g_2$.*

With this data one can associate the following groupoid $\mathcal{G}$: the objects are the points of $M$ and the morphisms $x \to y$, $x, y \in M$ are given by $g \in \Gamma$ such that $g(x) = y$, with the composition given by the product in $\Gamma$. Let $\mathcal{A}$ denote the convolution algebra of this groupoid, i.e. the cross-product $C_0^\infty(M) \rtimes \Gamma$. Let $\Omega^* = (\Omega^*(M) \rtimes \Gamma, d)$ denote the differential graded algebra of forms on $\mathcal{G}$ with the convolution product, where the differential $d$ is the de Rham differential. We will use the usual cross-product notations $\omega U_g$ for the elements of this algebra, where $\omega \in \Omega^*(M)$, $g \in \Gamma$. Since $\Gamma$ is, in general a pseudogroup, we suppose that

$$\mathrm{supp}\,\omega \subset \mathrm{Dom}\, g \tag{5.1}$$

Fix a trivialization of $E$. The action of $\Gamma$ on the bundle defines then a homomorphism

$$h : \mathcal{G} \to GL_n(\mathbb{R}) \tag{5.2}$$

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Let $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ denote the differential graded Hopf algebra of the forms on $GL_n(\mathbb{R})$, with the product given exterior multiplication, coproduct, antipode and counit induced respectively by the product $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, inverse $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ and the inclusion $1 \to GL_n(\mathbb{R})$. The differential is given by the de Rham differential on forms.

We now show that the map (5.2) allows one to define an action of $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ on $\Omega^*$.

**Proposition 9.** *The map* $\mathcal{H}\left(GL_n(\mathbb{R})\right) \otimes \Omega^* \to \Omega^*$ *given by*

$$\pi(\alpha)(\omega) = h^*(\alpha)\omega \tag{5.3}$$

*where* $\alpha \in \mathcal{H}\left(GL_n(\mathbb{R})\right)$, $\omega \in \Omega^*$ *defines an action of the differential graded Hopf algebra* $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ *on the differential graded algebra* $\Omega^*$.

*Proof.* We have:

$$\pi(\alpha_1\alpha_2)(\omega) = h^*(\alpha_1\alpha_2)\omega = h^*(\alpha_1)h^*(\alpha_2)\omega = \pi(\alpha_1)\left(\pi(\alpha_2)(\omega)\right) \tag{5.4}$$

Next, if we write $\Delta\alpha = \sum_k \alpha_{(0)} \otimes \alpha_{(1)}$ we have:

$$\pi(\alpha)(\omega_0\omega_1)(g) = h^*(\alpha)(g)\omega_0\omega_1(g) = h^*(\alpha)(g) \sum_{g_0g_1=g} \omega_0(g_0)\omega_1^{g_0}(g_1) =$$

$$\sum_{g_0g_1=g} \sum_k h^*(\alpha_{(0)})(g_0)h^*(\alpha_{(1)})^{g_0}(g_1)\omega_0(g_0)\omega_1^{g_0}(g_1) =$$

$$\sum_{g_0g_1=g} \sum_k (-1)^{\deg\omega_0 \deg\alpha_{(0)}} h^*(\alpha_{(0)})(g_0)\omega_0(g_0)h^*(\alpha_{(2)})(g_1)\omega_1^{g_0}(g_1) =$$

$$\sum_k (-1)^{\deg\omega_0 \deg\alpha_{(0)}} \pi(\alpha_{(0)})(\omega_0)\pi(\alpha_{(1)})(\omega_1) \tag{5.5}$$

Also, if $M$ is compact the algebra $\Omega^*$ has a unit given by the function

$$e(g) = \begin{cases} 1 & \text{if } g \text{ is a unit} \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

Then

$$\pi(\alpha)(e) = h^*(\alpha)e = \varepsilon(\alpha)e \tag{5.7}$$

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Finally, we have

$$d(\pi(\alpha(\omega))) = d\left(h^*(\alpha)\omega\right) =$$
$$h^*(d\alpha)\omega + (-1)^{\deg\alpha}h^*(\alpha)d\omega = \pi((d\alpha))(\omega) + (-1)^{\deg\alpha}\pi(\alpha)(d\omega) \quad (5.8)$$

$$\square$$

We now have a natural inclusion $i : M \hookrightarrow \mathcal{G}$ as the space of units. Then we define a graded trace $\fint$ on $\Omega^*$ by

$$\fint \omega = \int_M i^*\omega \quad (5.9)$$

**Proposition 10.** *The graded trace $\fint$ is closed under the de Rham differential and is invariant under the action of $\mathcal{H}$, i.e.*

$$\fint d\omega = 0 \quad (5.10)$$

$$\fint \alpha(\omega) = \varepsilon(\alpha)\fint \omega \quad (5.11)$$

*Proof.* The first identity is clear, the second follows from the fact that $h \circ i : M \to GL_n(\mathbb{R})$ is a constant map, taking the value 1. $\square$

Hence we have a map $\mathcal{B}_q\left(\mathcal{H}\left(GL_n(\mathbb{R})\right), d\right) \to \mathcal{B}\left(\Omega^*, d\right)$ where $q = \dim M$, which also gives us a map

$$\chi : HC_q^*\left(\mathcal{H}\left(GL_n(\mathbb{R})\right), d\right) \to HP^*\left(C_0^\infty(M) \rtimes \Gamma\right) \quad (5.12)$$

Definition of the action of $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ on $\Omega^*$, and hence definitions of the map given by (5.12) apriori depends on the choice of trivialization of $E$, but we will now show that this map is independent of the choice of trivialization.

**Proposition 11.** *Suppose we use another trivialization of $E$ to define an action of $\mathcal{H}\left(GL_n(\mathbb{R})\right)$ on $\Omega^*$. Then the two actions are conjugated by the inner automorphisms.*

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*Proof.* Let us chose another trivialization of the bundle $E$, and let $U(x)$ $x \in M$ be a transition matrix between the two bases of the fiber $E_x$. Then we have a new map $h' : \mathcal{G} \to GL_n(\mathbb{R})$, related to $h$ by

$$h'(\gamma) = U(s(\gamma))h(\gamma)U^{-1}(r(\gamma)) \tag{5.13}$$

Let $\pi'$ denote the action corresponding to the map $h'$.

Consider now the pull-back $U^* : \Omega^*(GL_n(\mathbb{R})) \to \Omega^*(M)$ as a map $\mathcal{H}(GL_n(\mathbb{R})) \to \Omega^*$, where we consider forms on $M$ as the form on $\mathcal{G}$ which is 0 outside the space of units. When $M$ is not compact, we obtain not an element in algebra, but rather a multiplier. Hence we see that if we define

$$\rho^+(\alpha) = U^*(\alpha) \tag{5.14}$$

and

$$\rho^-(\alpha) = (U^{-1})^*(\alpha) = \rho^+(S\alpha) \tag{5.15}$$

we will have

$$\pi'(\alpha)(\omega) = \sum (-1)^{\deg \alpha_{(2)} \deg \omega} \rho^+(\alpha_{(0)}) \pi(\alpha_{(1)})(\omega) \rho^-(\alpha_{(2)}) \tag{5.16}$$

$$\square$$

We can now summarize the results as follows.

**Theorem 12.** *Let $\Gamma$ be a discrete pseudogroup acting on the manifold $M$ of dimension $q$ by orientation preserving diffeomorphisms. Let $E$ be a $\Gamma$-equivariant trivial bundle of rank $n$ on $M$. Let $\mathcal{H}(GL_n(\mathbb{R}))$ be the DG Hopf algebra of the differential forms on the Lie group $GL_n(\mathbb{R})$. Then we have a map*

$$\chi : HC_q^i(\mathcal{H}(GL_n(\mathbb{R})), d) \to HP^{i-q}(C_0^\infty(M) \rtimes \Gamma) \tag{5.17}$$

*which is independent of the trivialization of $E$. In conjunction with the equation (6.1) below it gives a map*

$$\bigoplus_{m \in \mathbb{Z}} H^{i-2m}(W(\mathfrak{gl}_\mathfrak{n}, O_n)_q) \to HP^{i-q}(C_0^\infty(M) \rtimes \Gamma) \tag{5.18}$$

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# 6 Relation with Weil algebras

In this section we use methods of [12], [14] and [7, 8] to identify the cyclic cohomology $HC^*(\mathcal{H}\left(GL_n(\mathbb{R})\right), d)_q$. It turns out that

$$HC^i(\mathcal{H}\left(GL_n(\mathbb{R})\right), d)_q = \bigoplus_{m \geq 0} H^{i-2m}\left(W(\mathfrak{gl}_\mathfrak{n}, O_n)_q\right). \qquad (6.1)$$

where $H^*\left(W(\mathfrak{gl}_\mathfrak{n}, O_n)_q\right)$ is the cohomology of truncated Weil algebra (cf. [12]). As a matter of fact, one can work with the DG Hopf algebra $\mathcal{H}(G)$ of differential forms on any almost connected Lie group, and the result in this case is

$$HC^i(\mathcal{H}(G), d)_q = \bigoplus_{m \geq 0} H^{i-2m}\left(W(\mathfrak{g}, K)_q\right). \qquad (6.2)$$

where $K$ is the maximal compact subgroup of $G$. Computation of the Hochschild cohomology of this Hopf algebra is essentially contained in [12], and with little care using ideas from [7, 8] one recovers the cyclic cohomology.

Let $G$ be a Lie group with finitely many connected components. Similarly to the previous section we can define a differential graded Hopf algebra $\mathcal{H}(G)$. Let $K$ be the maximal compact subgroup of $G$. We will now construct the map of complexes from the truncated relative Weil algebra $W(\mathfrak{g}, K)_q$ to the complex $\mathcal{B}_q\left(\mathcal{H}(G)\right)$.

Let $NG$ denote the simplicial manifold with $NG_p = \underbrace{G \times G \times \ldots G}_{p}$ The simplicial structure is given by the face maps

$$\partial_i(g_1, g_2, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k) & \text{if } i = 0 \\ (g_1, g_2, \ldots g_i g_{i+1}, \ldots, g_k) & \text{if } 1 \leq i \leq k-1 \\ (g_1, \ldots, g_{k-1}) & \text{if } i = k \end{cases} \qquad (6.3)$$

and degeneracy maps

$$\sigma_i(g_1, g_2, \ldots, g_k) = (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_k) \qquad (6.4)$$

The geometric realization of this simplicial manifold is the classifying space B$G$. It is a union of manifolds $\Delta^p \times NG_p$ with the modulo the equivalence relation (cf. [8]).

We will also consider simplicial manifold $\bar{N}G$, with $\bar{N}G_p = \underbrace{G \times G \times \cdots \times G}_{p+1}$.
The face and degeneracy maps are given by

$$\partial_i(g_0, g_1, g_2, \ldots, g_k) = (g_0, \ldots \hat{g}_i, \ldots, g_k) \tag{6.5}$$

$$\sigma_i(g_0, g_1, g_2, \ldots, g_k) = (g_0, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, g_k) \tag{6.6}$$

The geometric realization of this simplicial manifold is EG. The map $pr :$ $\bar{N}G \to NG$ given by

$$pr(g_0, g_1, \ldots, g_p) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{p-1} g_p^{-1}) \tag{6.7}$$

defines a simplicial principal $G$-bundle EG $\to$ BG. Simplicial manifolds $\bar{N}G$ and $NG$ moreover have a cyclic structure, i.e. an action of the cyclic groups $\mathbb{Z}_{p+1}$ on the $p$-th component, which satisfy all the necessary relations with the face and degeneracy maps. The actions are given on $\bar{N}G$ by

$$\tau_p(g_0, g_1, \ldots, g_p) = (g_1, g_2, \ldots, g_p, g_0) \tag{6.8}$$

Since the maps $\tau_p$ are $G$-equivariant, they induce corresponding actions on $NG$:

$$\tau_p(g_1, g_2, \ldots, g_p) = (g_2, g_3, \ldots, g_p, (g_1 g_2 \ldots g_p)^{-1}) \tag{6.9}$$

We will identify the $p$-cochains for the Hopf algebra $\mathcal{H}(G)$ with the forms on $NG_p$. Under this identification the simplicial structure on the Hopf cochains corresponds to the one induced by the simplicial structure on $NG$, the de Rham differential on the Hopf cochains corresponds to the de Rham differential on $NG$, and cyclic structure on the Hopf cochains is induced by the cyclic structure on $NG$. Filtration by the form degree on the Hopf algebra cochains corresponds to the filtration by the form degree on the manifold $NG$.

We will now construct the map $\mu$ from the complex $W(\mathfrak{g}, K)$ to the simplicial-de Rham complex of forms on $NG$, which preserves filtration on these complexes. We do it by constructing the map from $W(\mathfrak{g}, K)$ to the complex of simplicial forms on B$G$, and then applying the integration map. The complex of simplicial forms on B$G$ has a natural bigrading. Let $\theta$ be the Maurer-Cartan form on $G$. Let $p_i : \bar{N}G = G^{p+1} \to G$ be the projection on $i$-th component. Consider on EG$_\mathrm{p}$ the $\mathfrak{g}$ valued differential form

$\omega \sum t_i \theta_i$, where $\theta_i = p_i^* \theta$. It defines a simplicial connection in the bundle EG $\to$ BG. The standard construction defines a differential graded algebra homomorphism $\psi$ from $W(\mathfrak{g}, K)$ to the complex of $K$-basic simplicial forms on EG, which we identify with forms on the space EG/K. This space is a bundle over B$G$ with the fiber $G/K$. This bundle has a section, which can be explicitly described as follows. Since $G/K$ has a natural structure of a manifold of constant negative curvature, for any finite set of points $x_0$, $x_1$, ..., $x_k \in G/K$ one can construct a canonical simplex in $G/K$, i.e. a map $\sigma(x_0, x_1, \ldots, x_k) : \Delta^k \to G/K$, with vertices $x_0$, $x_1$, ..., $x_k$, and this construction agrees with taking faces of a simplex, and is $G$-equivariant:

$$\sigma(gx_0, gx_1, \ldots, gx_k)(t_0, t_1, \ldots, t_k) = g\sigma(x_0, x_1, \ldots, x_k)(t_0, t_1, \ldots, t_k) \quad (6.10)$$

Denote by $\pi$ the canonical projection $G \to G/K$. Then the section $s$ is given by the simplicial map defined by the following formula, where we write just $\sigma$ for $\sigma(\pi(1), \pi(g_1), \pi(g_1 g_2), \ldots \pi(g_1 \ldots g_k)(t_0, t_1, \ldots, t_k)$:

$$s(g_1, g_2, \ldots, g_k; t_0, t_1, \ldots t_k) =$$
$$\left( \sigma^{-1}, g_1 \sigma^{-1}, g_1 g_2 \sigma^{-1}, \ldots g_1 \ldots g_k \sigma^{-1}; t_0, t_1, \ldots t_k \right) \quad (6.11)$$

Notice that this section intertwines the actions of the cyclic group on the spaces B$G$ and EG/K.

Consider now the map $s^* \psi$. It is clearly a homomorphism from the differential graded algebra $W(\mathfrak{g}, K)$ to the differential graded algebra of simplicial forms on B$G$. We will show now that it preserves filtration's on both algebras. First we need the following statement:

**Lemma 13.** *If $\xi$ is a horizontal form on* EG *of the type* $(k, l)$,*then $s^* \xi$ is also of the type* $(k, l)$.

*Proof.* Since $\xi$ is horizontal, it can be written as a sum of the expressions of the form $f pr^* \zeta$, where $f$ is a function, $pr : $ EG $\to$ BG – projection, and $\zeta$ is a form on B$G$ of the type $(k, l)$. Then $s^* (f pr^* \zeta) = (s^* f) \zeta$ is also of the type $(k, l)$. $\square$

**Proposition 14.** *The homomorphism $s^* \psi$ agrees with filtrations on the Weil algebra and on the forms on* B$G$.

*Proof.* The curvature of the connection $\omega$ is a horizontal form $\Omega$ on B$G$, given by

$$\Omega = \sum dt_i \theta_i + \sum t_i d\theta_i + \sum_{i<j} t_i t_j [\theta_i, \theta_j] \qquad (6.12)$$

Hence $\Omega$ has only components of the type $(1,1)$ and $(0,2)$. The statement of the lemma will then follow from the fact that $s^*\Omega$ also has only components of the type $(1,1)$ and $(0,2)$. But this follows from the Lemma 13. $\square$

We can now apply the integration map and obtain the map $\mu$ from the Weil algebra to the simplicial-de Rham complex of $NG$. Since the integration map respects filtrations, the resulting map $\mu$ also respects filtrations. We identify the Hochschild complex of $\mathcal{H}(G)$ with the simplicial-de Rham complex of $NG$. Results of [12], [14] imply that this is actually an isomorphism, i.e.

$$HH^i(\mathcal{H}(G), d)_q = H^i\left(W(\mathfrak{g}, K)_q\right) \qquad (6.13)$$

But since the connection $\omega$ and the section $s$ are invariant under the cyclic action, the resulting Hochschild cochains are actually cyclic. This implies that Connes' long exact sequence is equivalent to the collection of short exact sequnces

$$0 \to HC^{i-2}(\mathcal{H}(G), d)_q \xrightarrow{S} HC^i(\mathcal{H}(G), d)_q \xrightarrow{I}$$
$$\xrightarrow{I} HH^i(\mathcal{H}(G), d)_q \to 0 \quad (6.14)$$

and the map $I$ splits. Hence

$$HC^i(\mathcal{H}(G), d)_q =$$
$$\bigoplus_{m\geq 0} HH^{i-2m}(\mathcal{H}(G), d)_q = \bigoplus_{m\geq 0} H^{i-2m}\left(W(\mathfrak{g}, K)_q\right). \quad (6.15)$$

Explicitly, maps $H^{i-2m}\left(W(\mathfrak{g}, K)_q\right) \to HC^i(\mathcal{H}(G), d)_q$ are given by $S^m \circ \mu$, where we consider $\mu$ as a map into cyclic complex.

# 7 Relation with other constructions

Suppose, as before, that we have an orientation-preserving action of discrete group $\Gamma$ on an oriented manifold $M$, and an equivariant trivial bundle $E$ over $M$. Then results of previous sections provide us a map $H^*(W(\mathfrak{g}, O_n)) \to HP^*(\mathcal{A})$, where $\mathcal{A} = C_0^\infty(M) \rtimes \Gamma$. We also have a construction of the map $H^*(W(\mathfrak{g}, O_n)) \to H^*(M_\Gamma)$ (see e.g. [12, 1, 2]) where $M_\Gamma = M \times_\Gamma E\Gamma$ is the homotopy quotient. In this section we prove that these constructions are compatible, i.e. that the following diagram is commutative

$$
\begin{array}{ccc}
H^*(W(\mathfrak{g}, O_n)) & \longrightarrow & H^*(M_\Gamma) \\
& \searrow & \uparrow{\scriptstyle \Phi} \\
& & HP^*(\mathcal{A})
\end{array}
\qquad (7.1)
$$

where $\Phi$ is the canonical map given by Connes [3, 4].

The proof goes as follows. We construct a map $\Psi$ from the complex computing $H^*(M_\Gamma)$ to the cyclic complex $\mathcal{B}(\Omega^*, d)$, where $\Omega^* = \Omega_0^*(M) \rtimes \Gamma$, which has the following properties. First, it agrees with the map $\Phi$, in the sense that the following diagram is commutative:

$$
\begin{array}{ccc}
H^*(M_\Gamma) & \overset{\Phi}{\longrightarrow} & HP^*(\mathcal{A}) \\
& \searrow{\scriptstyle \Psi} & \uparrow{\scriptstyle R} \\
& & HP^*(\Omega^*, d)
\end{array}
\qquad (7.2)
$$

where the map $R$ is defined by (2.20). Then it is clear from the definitions that the diagram similar to (7.1) is valid with the map $\Psi$ already on the level of cochains, not just cohomology.

The definition of the map $\Psi$ is the following. Recall that the cohomology of $M_\Gamma$ can be computed by the following bicomplex $\mathcal{C}^{*,*}$. $\mathcal{C}^{k,l}$ denotes the set of totally antisymmetric functions $\tau$ on $\underbrace{\Gamma \times \Gamma \cdots \times \Gamma}_{k+1}$ with values in $-l$-currents on $\mathrm{Dom}\, g_0 \cap \mathrm{Dom}\, g_1 \cdots \cap \mathrm{Dom}\, g_k$, which satisfy the invariance condition

$$
\tau(g g_0, g g_1, \ldots, g g_k) = \tau(g_0, g_1, \ldots, g_k)^{g^{-1}}.
\qquad (7.3)
$$

The two differentials of this complex are given by the the group cohomol-

ogy complex differential given on $\mathcal{C}^{k,l}$ by

$$(d_1\tau)(g_0, g_1, \ldots, g_k, g_{k+1}) = (-1)^l \sum_{j=0}^{k+1} (-1)^j \tau(g_0, g_1, \ldots, \hat{g}_j, \ldots, g_{k+1}) \quad (7.4)$$

and the de Rham differential $d$ given by

$$(d_2\tau)(g_0, g_1 \ldots, g_k) = d\left(\tau(g_0, g_1, \ldots, g_k)\right) \quad (7.5)$$

We now define the map $\Psi$ from the complex $\mathcal{C}^{*,*}$ to the cyclic complex $\mathcal{B}(\Omega^*, d)$, where $\Omega^* = \Omega_0^*(M) \rtimes \Gamma$, by the following formula.

$$\Psi(\tau)(\omega_0 U_{g_0}, \omega_1 U_{g_1}, \ldots, \omega_k U_{g_k}) =$$
$$\begin{cases} (-1)^{kl} \langle \tau(1, g_0, g_0 g_1, \ldots, g_0 \ldots g_{k-1}), \omega_0 \omega_1^{g_0} \ldots \omega_k^{g_0 \ldots g_{k-1}} \rangle & \text{if } g_0 \ldots g_k = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$(7.6)$$

Map $\Psi$ satisfies the following identities

$$b\Psi(\tau) = \Psi(d_1\tau) \qquad (7.7)$$
$$d\Psi(\tau) = \Psi(d_2\tau) \qquad (7.8)$$
$$B\Psi(\tau) = 0 \qquad (7.9)$$

and hence it is indeed a map of complexes. It is clear from the definition of the map $\Psi$ that the diagram obtained from the diagram (7.1) by replacing $\Phi$ by $\Psi$ commutes, even on the level of complexes. It remains to prove that the map $\Psi$ induces the same map in cohomology as the map $\Phi$.

To do this we note that Connes' map $\Phi$ is characterized uniquely by the following properties (cf. [10, 11]):

1. $\Phi$ takes values in the part of cyclic cohomology supported at the identity of the group.

2. Let $M$ be an oriented manifold on which discrete group $\Gamma$ acts freely and properly preserving orientation. Then any class in $c \in H^*(M_\Gamma)$

can be represented by a $\Gamma$-invariant current $C$ on $M$. Then $\Phi(c)$ is the same as the class of the cyclic cocycle defined by

$$\phi(a_0 U_{g_0}, a_1 U_{g_1}, \ldots, a_k U_{g_k}) =$$
$$\begin{cases} \langle C, a_0 da_1^{g_0} \ldots da_k^{g_0 g_1 \ldots g_{k-1}} \rangle & \text{if } g_0 g_1 \ldots g_k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.10}$$

3. Let $X$ be another oriented $\Gamma$-manifold. Then one has the following commutative diagram

$$H^*\left((M \times X)_\Gamma\right) \xrightarrow{\Phi} HP^*\left(C_0^\infty(M \times X) \rtimes \Gamma\right) \tag{7.11}$$
$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$
$$H^*(M_\Gamma) \xrightarrow{\qquad \Phi \qquad} HP^*\left(C_0^\infty(M) \rtimes \Gamma\right)$$

The left vertical arrow here is induced by the natural map $(M \times X)_\Gamma \to M_\Gamma$ and the right one is induced by the product with the transverse fundamental class of $X$.

It is easy to see that the map $\Psi$ satisfies the same properties, and hence gives the same map in cohomology.

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